CERTAIN EXACT SOLUTIONS IN THE THEORY OF ELASTIC PLATES

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The Berger equations [1] have exact solutions in a number of cases [2]. However, it has been believed until recently that those solutions are special approximations of solutions of the Föppl–Kármán equations [3-6]. In the present paper it is shown that for a square plate for a centrally symmetric stress–strain state the solutions of Berger equations are identical with the solutions of the Föppl–Kármán equations.

1. The equations of bending of a rectangular ($0 \leqslant x_1 \leqslant a_1,\ 0 \leqslant x_2 \leqslant a_2$) plate obtained on the basis of the "Berger hypothesis" (neglect of the influence of the second invariant of the deformation tensor J_2 on the stress–strain state of the system) have the form

$$\partial J_1/\partial x_1 = 0,\ \partial J_1/\partial x_2 = 0; \tag{1.1}$$

$$D\nabla^2\nabla^2 w - 12Dh^{-2}J_1\nabla^2 w = q(x_1,\ x_2), \tag{1.2}$$

$$J_1 = \frac{1}{a_1a_2}\int\limits_0^{a_2}\int\limits_0^{a_1}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right]dx_1dx_2.$$

Here $J_1 = \varepsilon_{11} + \varepsilon_{22}$; $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$; ϵ_{11}, ϵ_{22}, and ϵ_{12} are deformations: expansion–compression and displacement, respectively; u_1, u_2, w are the displacements of points of the middle surface; $D = Eh^3/[12(1-\nu^2)]$; E, ν are the Young's modulus and the Poisson coefficient of the plate material; h is its thickness.

We write down the expression for this part of deformation energy of the middle surface of the plate which depends on the second invariant of the deformation tensor $J_2(J_2 = \varepsilon_{11}\varepsilon_{22} - 0.25\varepsilon_{12}^2)$:

$$\Pi = -D(1-\nu)\int\limits_0^{a_2}\int\limits_0^{a_1} J_2 dx_1dx_2 = -D(1-\nu)\int\limits_0^{a_2}\int\limits_0^{a_1}\left\{\left[\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{1}{2}\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{1}{4}\left(\frac{\partial u_1}{\partial x_2}\right)^2 - \frac{1}{4}\left(\frac{\partial u_2}{\partial x_1}\right)^2\right] - \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2} - \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1}\left(\frac{\partial w}{\partial x_2}\right)^2 + \frac{\partial u_2}{\partial x_2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]\right\}dx_1dx_2. \tag{1.3}$$

It is easy to convince oneself that for a square plate ($a_1 = a_2 = a$) in case of a centrally symmetric stress–strain state ($u_1 \rightleftarrows u_2$, $x_1 \rightleftarrows x_2$) $\Pi = 0$. the Berger equations give exact solutions of the Föppl–Kármán equations. This fact, to the authors' knowledge, has not been acknowledged in the literature. It demonstrates, for instance, that the solution of the nonlinear problem of bending of a plate supported by tilt bearing on its edges under a concentrated force q_0 is an exact solution of the Föppl–Kármán equations.

Indeed, if the plate is supported on a rigid undeformable contour, which prevents the edges from drawing closer together (for $x_1 = 0$, a $u_1 = u_2 = w = w_{x_1x_1} = 0$, for $x_2 = 0,a$ $u_1 = u_2 = w = w_{x_2x_2} = 0$), the expression for J_1 is simplified:

$$J_1 = \frac{1}{2a^2}\int\limits_0^a\int\limits_0^a\left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2\right]dx_1dx_2. \tag{1.4}$$

We will represent the load in the form of a double trigonometric series

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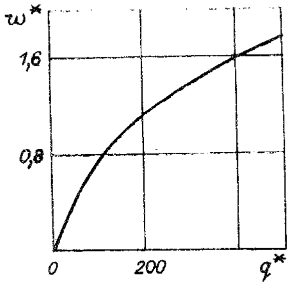

Fig. 1

$$q_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{a}, \quad c_{mn} = 4q_0 a^{-2} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}, \tag{1.5}$$

and the deflection w will be sought in the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{a}. \tag{1.6}$$

Substituting the expressions (1.5), (1.6) into (1.1), (1.4) and splitting the resulting relations according to sine functions, we obtain the following system of coupled nonlinear equations for the coefficients f_{mn}: $(m^2+n^2)^2 f_{mn} + 1.5h^{-2} f_{mn}(m^2+n^2) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij}(i^2+j^2) = a^4 \pi^{-4} D^{-1} c_{mn}$, $m, n = 1, 3, 5, \ldots,$ or

$$f_{mn} = \frac{a^4 c_{mn}}{\pi^4 D (m^2+n^2)(m^2+n^2+\lambda)}, \quad m, n = 1, 3, 5, \ldots,$$
$$\lambda = 1.5h^{-2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij}^2 (i^2+j^2). \tag{1.7}$$

Summing the relations (1.7) we find

$$\lambda = 24 \left(\frac{q_0 a^2}{\pi^4 D h} \right)^2 \sum_{m=1,3,\ldots} \sum_{n=1,3,\ldots} \frac{1}{(m^2+n^2)(m^2+n^2+\lambda)^2}. \tag{1.8}$$

In this manner the solution of the problem is reduced to a solution of the transcendental equation (1.8) and the linear system (1.7) with respect to the coefficients f_{mn}. In Fig. 1 we show the dependence of the deflection ($w^* = wh^{-1}$) in the center of the plate on the concentrated load acting at that point ($q^* = q_0 a^2 (Dh)^{-1}$).

2. In the dynamic case, we convince ourselves that the study of the free nonlinear vibrations of the plate of centrally symmetric form by using the Berger relations (1.1), (1.4) (for $q = -\rho h \partial^2 w / \partial t^2$) gives the possibility of obtaining an exact solution of the dynamical Föppl–Kármán equations. For the tilt bearing support of the plate on its contour the space and time variables in Eqs. (1.1), (1.4) are separated exactly and it becomes easy to determine the normal form of vibrations [2]: $w_{mn} = f_{mn} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{a} \operatorname{cn}\left(\frac{2\pi^2}{a^2} \sqrt{\frac{D}{\rho}(1+\gamma)}\, t, \sqrt{\frac{0.5\gamma}{1+\gamma}} \right)$, $\gamma = 1.5 (f_{mn}/h)^2$.

In connection with the presented notions it becomes rather doubtful whether an attempt to estimate the accuracy of the dynamicsl Berger equations could be successful when it is based on an approximate (found by the Bubnov–Galerkin method) solution of the Föppl–Kármán equations [6]. One must also recognize as unjustified the statement of the authors of [5] saying that the Berger equations give a larger error in studies of the dynamics of systems, including the centrally symmetric forms of vibrations of a square plate.

3. In the study of stability loss of a square plate of symmetric form we reached the following results. If compression forces act in the plane of the plate $N_1 = N_2 = -N$, then the Berger equations, which yield the exact solution of the Föppl–Kármán equations, become linearized: $\nabla^2 \nabla^2 w + (12N/Bh^2) \nabla^2 w = 0$, $B = 2Eh/(1-\nu^2)$. In case when on the contour of the

plate a "dead" load is acting (for $x_1 = 0$ and $u_1 = U$, $u_2 = w = w_{x1} = 0$, for $x_2 = 0$, and $u_2 = U$, $u_1 = w = w_{x2} = 0$), in order to obtain the exact solution one has to use equations of the form $\nabla^2\nabla^2 w + \frac{6}{h^2a^2}\left[\int_0^a\int_0^a\left\{\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2\right\}dx_1dx_2 + 8Ua\right]\nabla^2 w = 0.$

We note that the Berger equations offer the possibility to obtain exact solutions in the case of study of the centrally symmetric stress–strain state and the dynamics of a square plate in other important situations as well: plate on an elastic base (a solid one or centrally symmetric discrete) of centrally symmetric variable thickness, in particular, reinforced in two principal directions by a regular symmetric force (if only the bending rigidity of the ribbing is taken into account).

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